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Note

The Average Size of an Independent Set in Graphs with a Given Chromatic Number

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Let G be a graph on n vertices, let $\chi(G)$ denote its chromatic number, and let $\alpha(G)$ denote the average size of an independent set of vertices of G. A simple argument shows that $\alpha(G) = \Omega(n/\chi G) \log \chi(G))$. Here we show that this is sharp by constructing, for every $\chi \leqslant O(n/\log n)$, a graph G with $\chi(G) = \chi$ and with $\alpha(G) = \Theta(n/\chi \log \chi)$. This settles a problem of Linial and Saks.

Let G be a graph on n vertices, let $\chi = \chi(G)$ denote its chromatic number, and let $\alpha = \alpha(G)$ denote the average size of an independent set of vertices of G. A simple argument, due to N. Alon (cf. [LS]), shows that

$$\alpha = \Omega(n/\chi \log \chi). \tag{1}$$

Indeed, G must have an independent set of size at least n/χ, and hence it has $2^{n/\chi}$ subsets of it whose average size is $n/2\chi$. On the other hand, standard estimates imply that the total number of subsets of cardinality not exceeding $n/10\, \chi \log \chi$ of an n element set does not exceed $2^{n/\chi}$. Therefore, even if all these small subsets of vertices are independent, still the average size of an independent set in G satisfies (1).

Linial and Saks [LS] asked if (1) can be improved to

$$\alpha = \Omega(n/\chi).$$

The following theorem shows that this is false, and that (1) is sharp.

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THEOREM. (i) *For every* $\chi \leqslant 10\, n/\log n$ *there exists a graph* G *on* n *vertices which satisfies* $\chi(G) = \chi$ *and* $\alpha(G) = O(n/\chi \log \chi)$ (*and hence, by* (1), $\alpha(G) = \Theta(n/\chi \log \chi)$).

(ii) *For every* $\chi \geqslant 10\, n/\log n$ *there exists a graph* G *on* n *vertices which satisfies* $\chi(G) = \chi$ *and* $\alpha(G) = 2 + o(1)$.

Proof. First observe that it is enough to prove (ii) for $\chi = 10\, n/\log n$. Indeed, if we take a disjoint union of several copies of a graph G, we get a graph with the same chromatic number and with the same value of the quantity $\alpha/\{\textit{number of vertices}\}$. Thus (ii) for $\chi = 10n/\log n$ implies (i), and trivially implies (ii) for $\chi > 10n/\log n$. (Simply add, successively, new vertices joined to all other vertices to increase the chromatic number without increasing α.)

To prove (ii) for $\chi = 10n/\log n$ we construct the following graph G. Let $H = (V, E)$ be a graph on n vertices, with $\Theta(n^{3/2})$ edges and with girth at least 5. (Such graphs exist; e.g., take the points versus lines incidence graph of a projective geometry with m points (cf., e.g., [B]), where $n/4 < m < n/2$, and add some isolated vertices to make the number of vertices n.) Put $k = \lceil 10n/\log n \rceil$ and let $V = V_1 \cup V_2 \cup \cdots \cup V_k$ be an arbitrary partition of V into k pairwise disjoint sets, each of cardinality at most $\lceil \log n/10 \rceil$. Let T be the graph obtained from H by adding to H all edges (u, v), where $u, v \in V_i$ for some $1 \leqslant i \leqslant k$. Finally, let G be the complement of T. Clearly $\chi(G) \leqslant k$. The number of independent sets of size 2 in G is $\Omega(n^{3/2})$. We claim that the number of all other independent sets is $O(n^{1.1}/\log n)$ and each of them is of size $O(\log n)$. From this it follows that $\alpha(G) = 2 + o(1)$, as needed. To estimate the number of independent sets of size at least 3 in G, observe that this is precisely the number of cliques of size at least 3 in T. Let W be the set of vertices of such a clique. Clearly W cannot intersect three distinct V_i's, since otherwise H contains a triangle, contradicting the fact that its girth is at least 5. Similarly, W cannot intersect two V_i's, each by at least two elements, since otherwise H contains a cycle of length 4. Thus, for each such W, there are indices i, j so that $|W \cap V_i| \leqslant 1$ and $W \subseteq V_i \cup V_j$. However, for each choice of $U = W \cap V_j$ there is only one choice, at most, for V_i and for an element $v \in V_i$ such that $\{v\} \cup U$ is a clique in T, since $|W \cap V_j| \geqslant 2$ and every pair of vertices of V_j has at most one common neighbour outside V_j, as H contains no cycle of length 4. The total number of choices for U is $k \cdot 2^{\lceil 1/10 \cdot \log n \rceil} = O(n^{1.1}/\log n)$ and each such U is of size $O(\log n)$. This establishes the claim. One can easily check that it is possible to add $o(n^{3/2})$ edges to G and make its chromatic number precisely k, keeping the estimate $\alpha = 2 + o(1)$. Indeed, at least half of the vertices of G are joined to all others but $O(n^{1/2})$. Hence $O(kn^{1/2}) = o(n^{3/2})$ extra edges suffice to generate a complete graph on k vertices in G. This completes the proof of the theorem. ∎

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References

[B] B. Bollobás, "Extremal Graph Theory," Academic Press, London/New York/San Francisco, 1978.

[LS] N. Linial and M. Saks, Every poset has a central element, *J. Combin. Theory Ser. A* **40** (1985), 195–210.